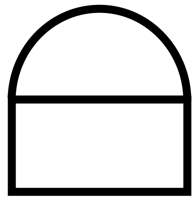
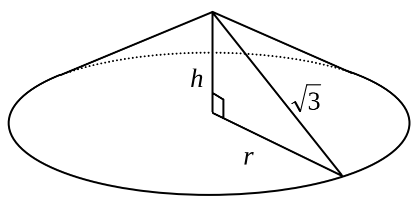
1) A Norman window has the outline of a semicircle on top of a rectangle as shown in the figure. Suppose there is $8+\pi$ feet of wood trim available for all 4 sides of the rectangle and the semicircle. Find the dimensions of the rectangle (and hence the semicircle) that will maximize the area of the window.



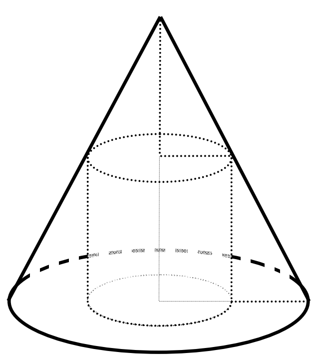
2) You are building a cylindrical barrel in which to put Dr. Brent so you can float him over Niagara Falls. I can fit in a barrel with volume equal 1 cubic meter. The material for the lateral surface costs \$18 per square meter. The material for the circular ends costs \$9 per square meter. What are the exact radius and height of the barrel so that cost is minimized?

3) A rectangular sheet of paper with perimeter 36 cm is to be rolled into a cylinder. What are the dimensions of the sheet that give the greatest volume?

4) A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume. **Note:** $V=\frac{1}{3}\pi r^2 h$.



5) Determine the cylinder with the largest volume that can be inscribed in a cone of height 8 cm and base radius 4 cm.



6) A straight piece of wire 8 feet long is bent into the shape of an L. What is the shortest possible distance between the ends?

7) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 9 - x^2$

8) A closed cylindrical container is to have a volume of 300 π in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$6 per in^2. Find the dimensions of the container of least cost.

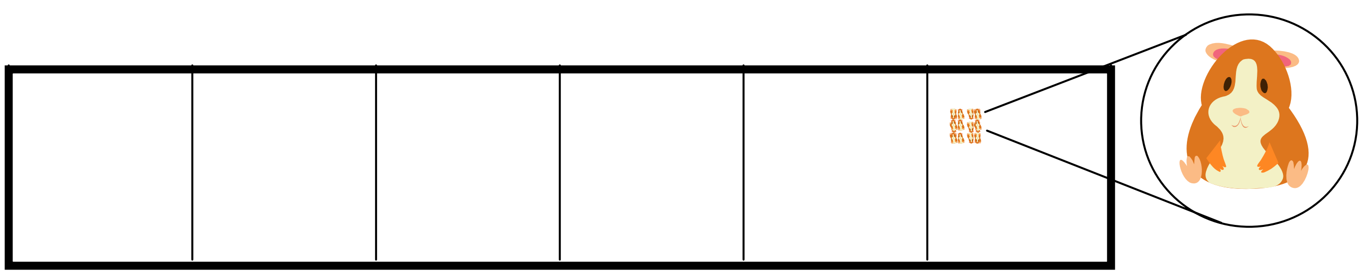
a)Draw a picture, label variables and write down a constrained optimization problem that models this problem. **(5 Pts)**

b) Using calculus, solve the problem in part (a) to find the dimensions.

9) A closed rectangular container with a square base is to have a volume of 300 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$6 per in^2. Find the dimensions of the container of least cost.

10) Your dream of becoming a hamster breeder has finally come true.

You are constructing a set of rectangular pens in which to breed your furry friends. The overall area you are working with is 60 square feet, and you want to divide the area up into six pens of equal size as shown below.



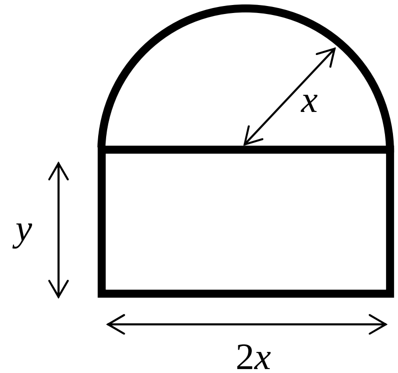
The cost of the outside fencing is \$10 a foot. The inside fencing costs \$5 a foot. You wish to minimize the cost of the fencing.

a) Labeling variables, write down a constrained optimization problem that describes this problem.

b) Using any method learned in this course, find the exact dimensions of each pen that will minimize the cost of the breeding ground. What is the total cost?

Solutions:

1) We will assume both x and y are positive, else we do not have the required window.



Let P be the wood trim, then the total amount is the perimeter of the rectangle $4x+2y$ plus half the circumference of a circle of radius x, or πx.

Hence the constraint is

$$P = 4x + 2y + \pi x = 8 + \pi$$

The objective function is the area

$$A = 2xy + \frac{1}{2}\pi x^2$$

Solving the constraint for y gives $y = \dfrac{8+\pi-(4+\pi)x}{2}$ and so

$$A = x(8+\pi-(4+\pi)x) + \frac{1}{2}\pi x^2$$

or

$$A = (8+\pi)x - (4+\pi)x^2 + \frac{1}{2}\pi x^2$$

And so we wish to maximize A over the interval $\left(0, \dfrac{8+\pi}{4+\pi}\right)$

$$\frac{dA}{dx} = (8+\pi) - 2(4+\pi)x + \pi x$$

Which is 0 when $x = 1$. Since $\dfrac{d^2A}{dx^2} = (-8-\pi) < 0$, we have indeed have a maximum.

Since $x = 1$ implies $y = 2$, the dimensions of the rectangle are 2 by 2 feet.

Student may choose the alternative way to solve the problem.

Assume y is a function of x and using implicit differentiation:

$$4x + 2y + \pi x = 8 + \pi \quad \text{and} \quad A = 2xy + \frac{1}{2}\pi x^2$$

$$\frac{d}{dx}(4x+2y+\pi x)=0 \quad \text{and} \quad \frac{dA}{dx}=\frac{d}{dx}(2xy+\frac{1}{2}\pi x^2)$$

or

$$4+2\frac{dy}{dx}+\pi=0 \quad \text{and} \quad \frac{dA}{dx}=2y+2x\frac{dy}{dx}+\pi x$$

so

$$\frac{dy}{dx}=-\frac{4+\pi}{2} \quad \text{and} \quad 2y+2x\frac{dy}{dx}+\pi x=0$$

$$2y-2x\frac{4+\pi}{2}+\pi x=0 \quad \text{or} \quad y=2x$$

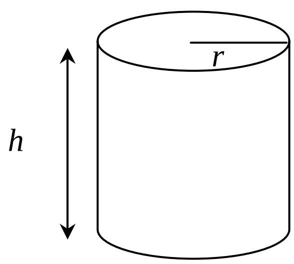
Using this in the original constraint eq. gives

$$4x+4x+\pi x = x(8+\pi)=8+\pi$$

So $x=1$, and $y=2$, and the dimension of the rectangle is 2 by 2 feet.

+=

2)



The volume is given by $V=\pi r^2 h$.
The circular ends have total area $2\pi r^2$.
The lateral surface area is $2\pi r h$.
Cost is $C=(\$18)(2\pi r h)+(\$9)(2\pi r^2)$.

So the problem at hand is minimize

$$C=36\pi r h+18\pi r^2$$

subject to the constraint $\pi r^2 h=1$.

$\pi r^2 h=1$ means $h=\dfrac{1}{\pi r^2}$. Using this here gives

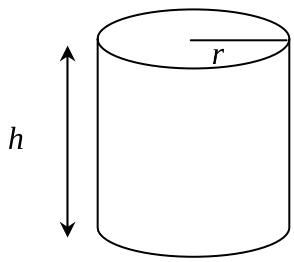
$$C=\frac{36}{r}+18\pi r^2$$

and so $$C'=-\frac{36}{r^2}+36\pi r\,,$$

which gives critical points as $r=0$, and $r=\dfrac{1}{\sqrt[3]{\pi}}$.

The physically reasonable solution is $r = \dfrac{1}{\sqrt[3]{\pi}}$ m which gives $h = \dfrac{1}{\sqrt[3]{\pi}}$ m.

Note: If you wish to solve the problem using implicit differentiation. The steps follow.



The volume is given by $V = \pi r^2 h$.
The circular ends have total area $2\pi r^2$.
The lateral surface area is $2\pi r h$.
Cost is $C = (\$18)(2\pi r h) + (\$9)(2\pi r^2)$.

So the problem at hand is minimize

$C = 36\pi r h + 18\pi r^2$

subject to the constraint $\pi r^2 h = 1$.

Assuming $h = f(r)$:

$$\frac{d}{dr}(\pi r^2 h) = \frac{d}{dr}(1) \quad \text{and} \quad \frac{d}{dr}C = \frac{d}{dr}(36\pi r h + 18\pi r^2)$$

$$2\pi r h + \pi r^2 \frac{dh}{dr} = 0 \quad \text{and} \quad \frac{dC}{dr} = 0 = 36\pi h + 36\pi r \frac{dh}{dr} + 36\pi r$$

$$\frac{dh}{dr} = -\frac{2h}{r}$$

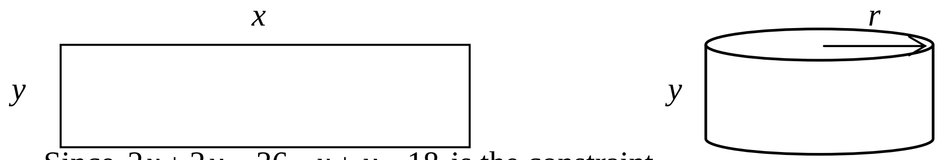
which gives

$$36\pi h + 36\pi r\left(-\frac{2h}{r}\right) + 36\pi r = 0 \text{ or } r = h.$$

Using $r = h$ in $\pi r^2 h = 1$ gives $r = h = \dfrac{1}{\sqrt[3]{\pi}}$ m

+=

3) Let x and y be the dimensions of the sheet of paper.

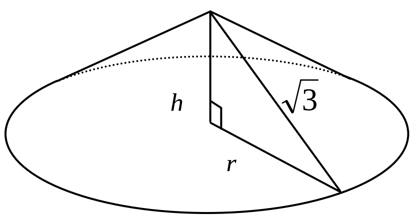


Since $2x+2y=36$, $x+y=18$ is the constraint.

The radius is given by $2\pi r = x$, so $r=\dfrac{x}{2\pi}$, and the volume is $V=\pi r^2 y=\dfrac{x^2 y}{4\pi}$. Using $y=18-x$, $V=\dfrac{x^2(18-x)}{4\pi}$ is the function to be optimized. $\dfrac{dV}{dt}=\dfrac{36x-3x^2}{4\pi}$, so critical numbers are $x=0,12$. Maximum volume occurs when $x=12$, (Why?) so dimensions are 6 cm by 12 cm and the volume is $\dfrac{216}{\pi}$ cm^3.

+=

4)



The volume of the cone is: $V=\dfrac{1}{3}\pi r^2 h$

The constraint equation is: $r^2+h^2=(\sqrt{3})^2$ or

$$r^2+h^2=3$$

1) Solving for r^2 gives $r^2=3-h^2$, so $V=\dfrac{1}{3}\pi(3-h^2)h$ or $V=\dfrac{1}{3}\pi(3h-h^3)$.

Taking the h-derivative of V gives $\dfrac{dV}{dh}=\dfrac{1}{3}\pi(3-3h^2)=\pi(1-h^2)$.

Stationary points are $h=\pm 1$, and the physically reasonable one is

$$h=1.$$

If $h=1$, then $r^2=3-h^2=3-1=2$, so $r=\sqrt{2}$, and the volume is $V=\dfrac{2}{3}\pi$.

2) Solving for h is ***FAR*** more difficult. $r^2+h^2=3$ means $h=\sqrt{3-r^2}$, and so $V=\dfrac{1}{3}\pi r^2\sqrt{3-r^2}$.

So $\dfrac{dV}{dr}=\dfrac{2}{3}\pi r\sqrt{3-r^2}+\dfrac{1}{3}\pi r^2\left(\dfrac{-r}{\sqrt{3-r^2}}\right)$

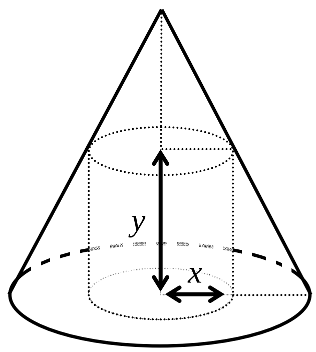
Rewriting this as

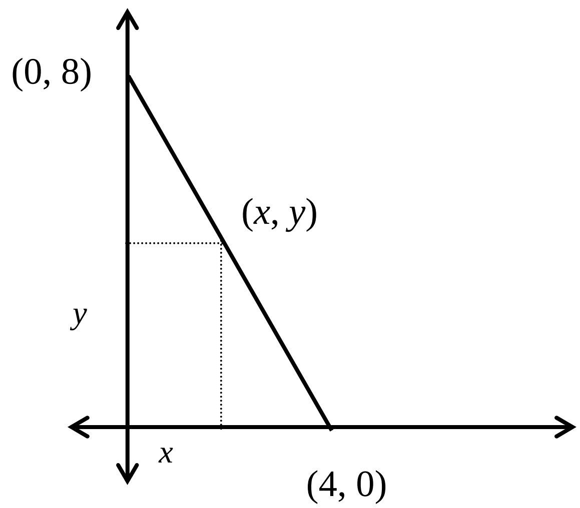
$$\frac{dV}{dr}=\frac{2}{3}\pi r\frac{3-r^2}{\sqrt{3-r^2}}+\frac{1}{3}\pi r^2\left(\frac{-r}{\sqrt{3-r^2}}\right)=\frac{\pi}{3}\frac{6r-2r^3-r^3}{\sqrt{3-r^2}}=\frac{r(2-r^2)}{\sqrt{3-r^2}}$$

So stationary points are $r=0,\pm\sqrt{2},\pm\sqrt{3}$. The only physically reasonable one is $r=\sqrt{2}$, so $h=\sqrt{3-2}=1$, and $V=\dfrac{2}{3}\pi$.

+=

5) Determine the cylinder with the largest volume that can be inscribed in a cone of height 8 cm and base radius 4 cm. **(15 points)**





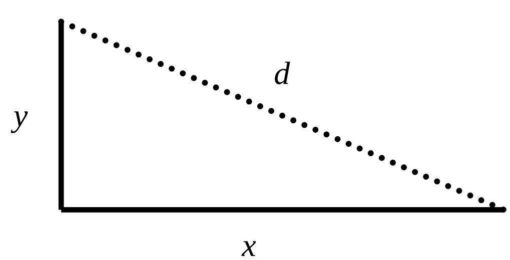
The volume of the cylinder is $V = \pi x^2 y$. The requirement that the cylinder is inscribed in the cone leads to the picture on the right. Since the edge of the cone is a straight line, we can use the two points (0, 8) and (4, 0) to determine the relation between x and y. Slope is $\frac{8-0}{0-4} = -2$. The line is $y = -2x + 8$.

So the volume becomes $V = \pi x^2 (8 - 2x) = \pi(8x^2 - 2x^3)$ and $x \in [0, 4]$.

$$V' = \pi(16x - 6x^2) = 2\pi x(8 - 3x)$$

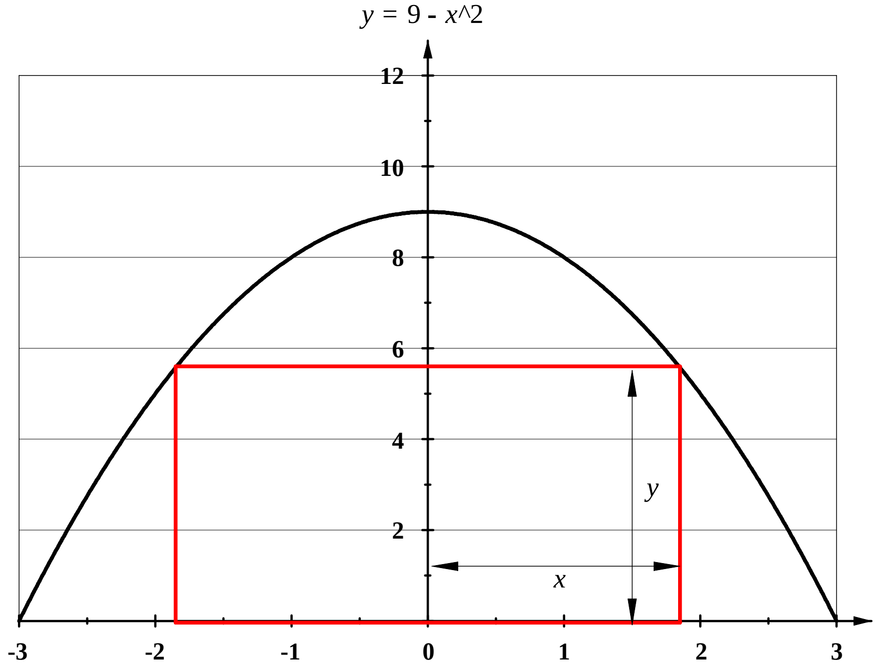
And so critical numbers are $x = 0, \frac{8}{3}$. The maximum occurs when the radius is $x = \frac{8}{3}$ which means the height is $y = \frac{8}{3}$, and the volume is $V = \frac{512\pi}{27}$ cubic cm.

6)A straight piece of wire 8 feet long is bent into the shape of an L. What is the shortest possible distance between the ends? **(20 Points)**



Let *d* be the distance between the ends of the L. then $d^2 = x^2 + y^2$. The constraint is $x + y = 8$, and so $y = 8 - x$, and then $d^2 = f(x) = x^2 + (8-x)^2$ which, when simplified, gives $f(x) = 2x^2 - 16x + 64$. Since *d* and d^2 have the same critical points we work with $f(x)$. To determine critical numbers we compute $f'(x) = 4x - 16$, and solve $4x - 16 = 0$, and so the critical number is $x = 4$, and so $y = 8 - x = 4$ also. The distance is then $d = \sqrt{x^2 + y^2} = \sqrt{32} = 4\sqrt{2}$. We know this is a minimum since $f''(x) = 4 > 0$.

7) Find the dimensions of the rectangle of largest area that has its base on the *x*-axis and its other two vertices above the *x*-axis and lying on the parabola $y = 9 - x^2$



The rectangle area is $A = 2x \cdot y$

The requirement that the rectangle lies on the graph of $y = 9 - x^2$ means

$$A = 2x \cdot (9 - x^2) = 18x - 2x^3$$

The variable *x* is restricted between [0,3] at which both points yield a minimum value of no rectangle or 0 area.

Since

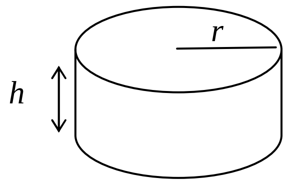
$$A'(x) = 18 - 6x^2 = 6(3 - x^2),$$

When

$$A'(x) = 6(3 - x^2) = 0 \text{ we get}$$

critical points at $x = \pm\sqrt{3}$. For our geometry we choose the positive root $x = \sqrt{3}$, and so, $y = 6$, so the dimensions are $2\sqrt{3}$ units by 6 units, and the area is $2xy = 12\sqrt{3}$ square units.

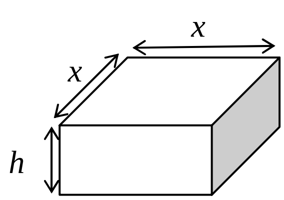
8)



Volume: $V = \pi r^2 h$ Cost: $\$2(2\pi r^2) + \$6(2\pi r h)$
So Problem is minimize cost $C = 4\pi r^2 + 12\pi r h$ subject to the constraint $V = \pi r^2 h = 300\pi$ and so $r^2 h = 300$.

b) Solving this last equation for h gives: $h = \dfrac{300}{r^2}$, which when substituted into the cost equation yields $C = 4\pi r^2 + \dfrac{3600\pi}{r}$. The geometry gives $r \in (0,\infty)$. To minimize the cost we determine critical numbers from $C' = 8\pi r - \dfrac{3600\pi}{r^2} = 0$ hence $r^3 = 450$ so the critical number is $r = (450)^{1/3}$. This gives $h = \dfrac{300}{(450)^{2/3}} = \dfrac{2(450)^{2/3}}{3}$ in. Since $C'' = 8\pi + \dfrac{7200}{r^3}\pi = 20\pi > 0$, the dimensions yield the minimum cost. The cylinder should have a radius $r = (450)^{1/3}$ in, and a height of $h = \dfrac{300}{(450)^{2/3}}$ in order to minimize the cost.

9) A closed rectangular container with a square base is to have a volume of 300 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$6 per in^2. Find the dimensions of the container of least cost. **(20 Points)**



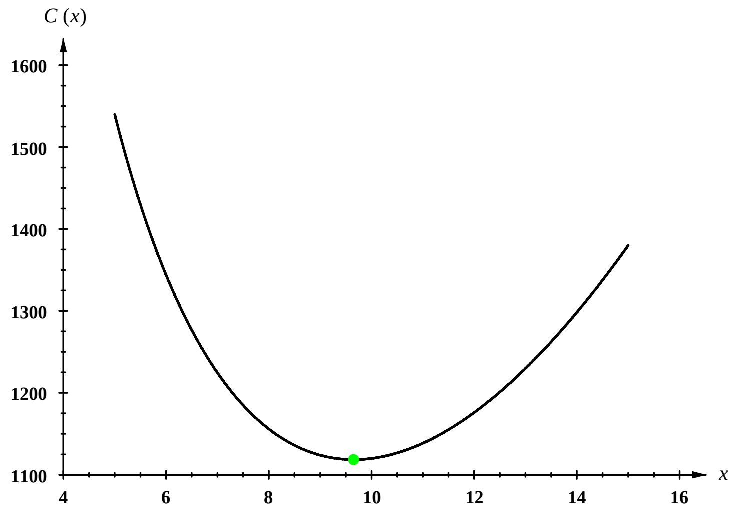
Volume: $V = x^2 h$ Cost: $\$2(2x^2) + \$6(4xh)$
So Problem is minimize $C = 4x^2 + 24xh$ subject to the constraint $x^2 h = 300$ Solving this last equation for h gives: $h = \dfrac{300}{x^2}$, which when substituted into the cost equation yields $C = 4x^2 + \dfrac{7200}{x}$. The geometry gives $x \in (0,\infty)$.

Since $C' = 8x - \dfrac{7200}{x^2} = 0$ gives $x^3 = 900$ whose solution is $x = 900^{1/3} \approx 9.65$. This gives $h = \dfrac{300}{900^{2/3}} = \dfrac{900^{1/3}}{3} \approx 3.22$ in. Note $x^2 h = 900^{2/3} \cdot \dfrac{900^{1/3}}{3} = \dfrac{900}{3} = 300\,\text{in}^3$ so the volume is

$$C = 4x^2 + \frac{7200}{x}$$

correct. And since $C'' = 8 + \frac{14{,}400}{x^3} = 8 + 16 > 0$, the dimensions yield the minimum cost. The box should have a square base of side length $900^{1/3}$ in, and a height of $\frac{900^{1/3}}{3}$ in.

FYI



10) The cost of the outside fencing is \$10 a foot. The inside fencing costs \$5 a foot. You wish to minimize the cost of the fencing.

a) Let x be the width of each individual pen, and y be the length as shown above. Since the total area is 60 sq. ft., each individual pen will have an area of 10 sq. ft.
The constraint is $xy = 10$. The objective function is the cost. Examining the fencing above, there is $5y$ feet of interior fencing, and $2y + 12x$ feet of exterior fencing.
So the total cost is $C = \$5 \cdot (5y) + \$10 \cdot (2y + 12x)$, or $C = 45y + 120x$.

The constrained optimization problem is: Minimize $C = 45y + 120x$ subject to the constraint $xy = 10$.

b) Solving $xy = 10$ of y gives $y = \frac{10}{x}$. Substituting this into C gives $C = \frac{450}{x} + 120x$, as the function to minimize over $x \in (0, \infty)$.

$C' = -\frac{450}{x^2} + 120$, and so critical points are $x = 0$, and $x = \frac{\sqrt{15}}{2}$, so

$y = \frac{10}{x} = \frac{4\sqrt{15}}{3}$.

The cost is $C = 45y + 120x = 120\sqrt{15} \approx \$464.76$.