

Extrema**Absolute Extrema**

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \leq f(x)$ for all x in the domain.

Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ then there exist numbers c and d so that,
 1. $a \leq c, d \leq b$, 2. $f(c)$ is the abs. max. in $[a, b]$, 3. $f(d)$ is the abs. min. in $[a, b]$.

Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a, b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a, b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min. (smallest function value) from the evaluations in Steps 2 & 3.

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b)

then there is a number $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Newton's Method

If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{\text{st}}$ guess is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

provided $f'(x_n)$ exists.

Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \geq f(x)$ for all x near c .
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \leq f(x)$ for all x near c .

1st Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

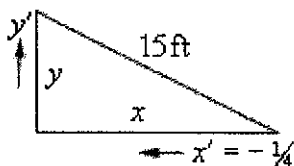
Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to t using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of t). Plug in known quantities and solve for the unknown quantity.

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,

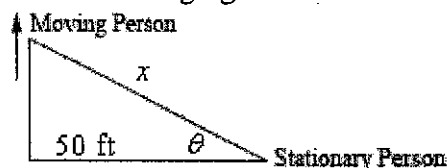
$$x^2 + y^2 = 15^2 \Rightarrow 2x x' + 2y y' = 0$$

After 12 sec we have $x = 10 - 12(\frac{1}{4}) = 7$ and

so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y' .

$$7(-\frac{1}{4}) + \sqrt{176} y' = 0 \Rightarrow y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

Ex. Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x' . We can use various trig fcn's but easiest is,

$$\sec \theta = \frac{x}{50} \Rightarrow \sec \theta \tan \theta \theta' = \frac{x'}{50}$$

We know $\theta = 0.5$ so plug in θ' and solve.

$$\sec(0.5) \tan(0.5)(0.01) = \frac{x'}{50}$$

$$x' = 0.3112 \text{ ft/sec}$$

Remember to have calculator in radians!

Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \Rightarrow A = y(500 - 2y) = 500y - 2y^2$$

Differentiate and find critical point(s).

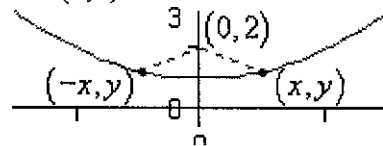
$$A' = 500 - 4y \Rightarrow y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find x .

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

Ex. Determine point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.



Minimize $f = d^2 = (x-0)^2 + (y-2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.

$$x^2 = y - 1 \Rightarrow f = x^2 + (y - 2)^2 = y - 1 + (y - 2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

$$f' = 2y - 3 \Rightarrow y = \frac{3}{2}$$

By the 2nd derivative test this is a rel. min. and so all we need to do is find x value(s).

$$x^2 = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

The 2 points are then $(\frac{1}{\sqrt{2}}, \frac{3}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{3}{2})$.