

Implicit Differentiation Selected Problems

Matthew Staley

September 20, 2011

Implicit Differentiation : Selected Problems

1. Find dy/dx .

$$\begin{aligned}
 \text{(a)} \quad y &= \sqrt[3]{2x - 5} \\
 &= (2x - 5)^{1/3} \\
 \frac{dy}{dx} &= \frac{1}{3}(2x - 5)^{1/3-1}(2) \\
 &= \boxed{\frac{2}{3}(2x - 5)^{-2/3}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad y &= \sqrt[3]{2 + \tan(x^2)} \\
 &= (2 + \tan(x^2))^{1/3} \\
 \frac{dy}{dx} &= \frac{1}{3}(2 + \tan(x^2))^{(1/3-1)}(\sec^2(x^2))(2x) \\
 &= \boxed{\frac{2}{3}x \sec^2(x^2)(2 + \tan(x^2))^{-2/3}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad y &= x^3(5x^2 + 1)^{-2/3} \\
 \frac{dy}{dx} &= x^3 \left(-\frac{2}{3}(5x^2 + 1)^{-5/3}(10x) \right) + 3x^2(5x^2 + 1)^{-2/3} \\
 &= \frac{-20x^4}{3(5x^2 + 1)^{5/3}} + \frac{3x^2}{(5x^2 + 1)^{2/3}} \\
 &= \frac{-20x^4}{3(5x^2 + 1)^{5/3}} + \frac{3x^2}{(5x^2 + 1)^{2/3}} \cdot \frac{3(5x^2 + 1)}{3(5x^2 + 1)} \\
 &= \frac{-20x^4}{3(5x^2 + 1)^{5/3}} + \frac{9x^2(5x^2 + 1)}{3(5x^2 + 1)^{5/3}} \\
 &= \frac{-20x^4 + 45x^4 + 9x^2}{3(5x^2 + 1)^{5/3}} \\
 &= \frac{25x^4 + 9x^2}{3(5x^2 + 1)^{5/3}} \\
 &= \boxed{\frac{1}{3}x^2(5x^2 + 1)^{-5/3}(25x^2 + 9)}
 \end{aligned}$$

2. Given $x + xy - 2x^3 = 2$.

(a) Find dy/dx by differentiating implicitly.

$$\begin{aligned}\frac{d}{dx}(x + xy - 2x^3) &= \frac{d}{dx}(2) \\ \frac{d}{dx}(x) + \frac{d}{dx}(xy) - \frac{d}{dx}(2x^3) &= 0 \\ 1 + x\frac{d}{dx}(y) + y - 6x^2 &= 0 \\ 1 + x\frac{dy}{dx} + y - 6x^2 &= 0\end{aligned}$$

(b) Solve the equation for y as a function of x , and find dy/dx from that equation.

$$\begin{aligned}x + xy - 2x^3 &= 2 \\ xy &= 2 + 2x^3 - x \\ y &= 2x^{-1} + 2x^2 - 1 \quad (*) \\ \frac{dy}{dx} &= -2x^{-2} + 4x\end{aligned}$$

(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

$$\begin{aligned}1 + x\frac{dy}{dx} + y - 6x^2 &= 0 \\ x\frac{dy}{dx} &= 6x^2 - y - 1 \\ \frac{dy}{dx} &= 6x - x^{-1}(y) - x^{-1} \quad \text{Sub (*) in part (b) into } y \text{ here.} \\ &= 6x - x^{-1}(2x^{-1} + 2x^2 - 1) - x^{-1} \\ &= 6x - 2x^{-2} - 2x + x^{-1} - x^{-1} \\ &= -2x^{-2} + 4x\end{aligned}$$

We see that (b) and (c) each give $\boxed{\frac{dy}{dx} = 4x - \frac{2}{x^2}}$.

3. Find dy/dx by implicit differentiation.

(a) $x^3 + y^3 = 3xy^2$

$$\begin{aligned}\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(3xy^2) \\ 3x^2 + 3y^2 \frac{dy}{dx} &= 3y^2 + 3x(2y) \frac{dy}{dx} \\ \frac{dy}{dx}(3y^2 - 6xy) &= 3y^2 - 3x^2 \\ \frac{dy}{dx} &= \frac{3y^2 - 3x^2}{3y^2 - 6xy} = \boxed{\frac{y^2 - x^2}{y^2 - 2xy}}\end{aligned}$$

(b) $x^2y + 3xy^3 - x = 3$

$$\begin{aligned}\frac{d}{dx}(x^2y + 3xy^3 - x) &= \frac{d}{dx}(3) \\ 2xy + x^2 \frac{dy}{dx} + 3y^3 + 3x(3y^2) \frac{dy}{dx} - 1 &= 0 \\ \frac{dy}{dx}(x^2 + 9xy^2) &= 1 - 2xy - 3y^3 \\ \frac{dy}{dx} &= \boxed{\frac{1 - 2xy + 3y^3}{x^2 + 9xy^2}}\end{aligned}$$

(c) $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$

$$\begin{aligned}\frac{d}{dx}(x^{-1/2} + y^{-1/2}) &= \frac{d}{dx}(1) \\ -\frac{1}{2}x^{-3/2} - \frac{1}{2}y^{-3/2} \frac{dy}{dx} &= 0 \\ -\frac{1}{2y^{3/2}} \frac{dy}{dx} &= \frac{1}{2x^{3/2}} \\ \frac{dy}{dx} &= \boxed{-\frac{y^{3/2}}{x^{3/2}}}\end{aligned}$$

$$(d) \quad \sin(x^2y^2) = x$$

$$\frac{d}{dx}(\sin(x^2y^2)) = \frac{d}{dx}(x)$$

$$\cos(x^2y^2) \frac{d}{dx}(x^2y^2) = 1$$

$$\cos(x^2y^2) \left(2xy^2 + x^2(2y) \frac{dy}{dx} \right) = 1$$

$$2xy^2 + 2x^2y \frac{dy}{dx} = \frac{1}{\cos(x^2y^2)}$$

$$2x^2y \frac{dy}{dx} = \frac{1}{\cos(x^2y^2)} - 2xy^2$$

$$\frac{dy}{dx} = \frac{1}{2x^2y \cos(x^2y^2)} - \frac{2xy^2}{2x^2y}$$

$$= \frac{1}{2x^2y \cos(x^2y^2)} - \frac{y}{x}$$

$$= \frac{1}{2x^2y \cos(x^2y^2)} - \frac{y}{x} \cdot \frac{2xy \cos(x^2y^2)}{2xy \cos(x^2y^2)}$$

$$= \boxed{\frac{1 - 2xy^2 \cos(x^2y^2)}{2x^2y \cos(x^2y^2)}}$$

4. Find d^2y/dx^2 by implicit differentiation.

(a) $2x^2 - 3y^2 = 4$

$$\frac{d}{dx}(2x^2 - 3y^2) = \frac{d}{dx}(4)$$

$$4x - 6y \frac{dy}{dx} = 0$$

$$-6y \frac{dy}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{2x}{3y}$$

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{2x}{3y} \right)$$

$$\frac{d^2y}{dx^2} = \frac{3y(2) - 2x(3) \frac{dy}{dx}}{(3y)^2}$$

$$= \frac{6y - 6x \left(\frac{2x}{3y} \right)}{9y^2}$$

$$= \frac{\left(\frac{2}{y} \right) (3y^2 - x^2)}{9y^2}$$

$$= \boxed{\frac{2(3y^2 - x^2)}{9y^3}}$$

(b) $xy + y^2 = 2$

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(2)$$

$$y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \quad (*)$$

$$\frac{dy}{dx}(x + 2y) = -y$$

$$\frac{dy}{dx} = -\frac{y}{x + 2y}$$

Now it is easier to differentiate (*) again instead of the above expression for $dy/dx \dots$

$$\frac{d}{dx} \left(y + x \frac{dy}{dx} + 2y \frac{dy}{dx} \right) = \frac{d}{dx} (0)$$

$$\frac{dy}{dx} + x \frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \cdot \frac{dy}{dx} = 0$$

$$2 \frac{dy}{dx} + \frac{d^2y}{dx^2} (x + 2y) + 2 \left(\frac{dy}{dx} \right)^2 = 0$$

$$\begin{aligned} \frac{d^2y}{dx^2} (x + 2y) &= -2 \left(\frac{dy}{dx} + \left(\frac{dy}{dx} \right)^2 \right) \\ &= -2 \left(-\frac{y}{x + 2y} + \left(-\frac{y}{x + 2y} \right)^2 \right) \\ &= -2 \left(-\frac{y}{x + 2y} + \frac{y^2}{(x + 2y)^2} \right) \\ &= -2 \left(-\frac{y(x + 2y) + y^2}{(x + 2y)^2} \right) \\ &= \frac{2xy + 4y^2 - 2y^2}{(x + 2y)^2} \end{aligned}$$

$$\frac{d^2y}{dx^2} (x + 2y) = \frac{2xy + 2y^2}{(x + 2y)^2}$$

$$\rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{2y(x + y)}{(x + 2y)^3}}$$

5. Use implicit differentiation to find the slope of the tangent line to the curve at the specified point.

$$(a) \quad x^4 + y^4 = 16; \quad (1, \sqrt[4]{15})$$

$$\begin{aligned} \frac{d}{dx}(x^4 + y^4) &= \frac{d}{dx}(16) \\ 4x^3 + 4y^3 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x^3}{y^3} \\ &= -\frac{(1)^3}{(\sqrt[4]{15})^3} \approx \boxed{-0.1312} \end{aligned}$$

$$(b) \quad 2(x^2 + y^2)^2 = 25(x^2 - y^2); \quad (3, 1)$$

$$\begin{aligned} \frac{d}{dx}(2(x^2 + y^2)^2) &= \frac{d}{dx}(25(x^2 - y^2)) \\ 4(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) &= 50x - 50y \frac{dy}{dx} \end{aligned}$$

Rather than try and solve for dy/dx now, we can just go ahead and plug in $x = 3$, $y = 1$ and solve for dy/dx with numbers which is much easier...

$$\begin{aligned} 4(3^2 + 1^2) \left(2(3) + 2(1) \frac{dy}{dx} \right) &= 50(3) - 50(1) \frac{dy}{dx} \\ 40 \left(6 + 2 \frac{dy}{dx} \right) &= 150 - 50 \frac{dy}{dx} \\ 240 + 80 \frac{dy}{dx} &= 150 - 50 \frac{dy}{dx} \\ 130 \frac{dy}{dx} &= -90 \\ \frac{dy}{dx} &= \boxed{-\frac{9}{13}} \end{aligned}$$