7.1.1 The Method

The method used in all the examples here can be summarized as follows:

1. Anticipate the form of the antiderivative by an *approximate form* (correct up to a multiplicative constant).
2. Differentiate this approximate form and compare to the original integrand function;
3. If Step 1 is correct, i.e., the approximate form's derivative differs from the original integrand function by a multiplicative constant, insert a compensating, reciprocal multiplicative constant into the approximate form to arrive at the actual antiderivative;
4. For verification, differentiate the answer to see if the original integrand function emerges.

For instance, some general formulas which should be quickly verifiable by inspection (that is, by reading and mental computation rather than with paper and pencil, for instance) follow:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \tag{7.1}$$

$$\int \cos kx\,dx = \frac{1}{k}\sin kx + C, \tag{7.2}$$

$$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C, \tag{7.3}$$

$$\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + C, \tag{7.4}$$

$$\int \csc^2 kx\,dx = -\frac{1}{k}\cot kx + C, \tag{7.5}$$

$$\int \sec kx \tan kx\,dx = \frac{1}{k}\sec kx + C, \tag{7.6}$$

$$\int \csc kx \cot kx\,dx = -\frac{1}{k}\csc kx + C, \tag{7.7}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C. \tag{7.8}$$

Example 7.1.2 *The following integrals can be computed with u-substitution, but also are computable by inspection:*

- $\displaystyle\int e^{7x}\,dx = \frac{1}{7}e^{7x} + C,$
- $\displaystyle\int \frac{1}{5x-9}\,dx = \frac{1}{5}\ln|5x-9| + C,$
- $\displaystyle\int \sin 5x\,dx = -\frac{1}{5}\cos 5x + C,$
- $\displaystyle\int \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2} + C,$
- $\displaystyle\int \sec^2 \pi x\,dx = \frac{1}{\pi}\tan \pi x + C,$
- $\displaystyle\int \csc 6x \cot 6x\,dx = -\frac{1}{6}\csc 6x + C.$

While it is true that we can call upon the formulas (7.1)–(7.8), the more flexible strategy is to anticipate the form of the antiderivative and adjust accordingly. For instance, we have the following antiderivative form, written two ways:

$$\int \frac{1}{u}\,du = \ln|u| + C,$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

"Power Reducers"
$\sin^2 x = \frac{1-\cos 2x}{2}$ $\cos^2 x = \frac{1+\cos 2x}{2}$

8.2 POWERS OF SINE AND COSINE

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

EXAMPLE 8.2.1 Evaluate $\int \sin^5 x\, dx$. Rewrite the function:

$$\int \sin^5 x\, dx = \int \sin x \sin^4 x\, dx = \int \sin x (\sin^2 x)^2\, dx = \int \sin x (1-\cos^2 x)^2\, dx.$$

Now use $u = \cos x$, $du = -\sin x\, dx$:

$$\begin{aligned}\int \sin x(1-\cos^2 x)^2\, dx &= \int -(1-u^2)^2\, du \\ &= \int -(1-2u^2+u^4)\, du \\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

□

EXAMPLE 8.2.2 Evaluate $\int \sin^6 x\, dx$. Use $\sin^2 x = (1-\cos(2x))/2$ to rewrite the function:

$$\begin{aligned}\int \sin^6 x\, dx = \int (\sin^2 x)^3\, dx &= \int \frac{(1-\cos 2x)^3}{8}\, dx \\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\, dx.\end{aligned}$$

Now we have four integrals to evaluate:

$$\int 1\, dx = x$$

and

$$\int -3\cos 2x\, dx = -\frac{3}{2}\sin 2x$$

are easy. The $\cos^3 2x$ integral is like the previous example:

$$\begin{aligned}\int -\cos^3 2x\,dx &= \int -\cos 2x \cos^2 2x\,dx\\ &= \int -\cos 2x(1-\sin^2 2x)\,dx\\ &= \int -\frac{1}{2}(1-u^2)\,du\\ &= -\frac{1}{2}\left(u-\frac{u^3}{3}\right)\\ &= -\frac{1}{2}\left(\sin 2x-\frac{\sin^3 2x}{3}\right).\end{aligned}$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x+\frac{\sin 4x}{4}\right).$$

So at long last we get

$$\int \sin^6 x\,dx = \frac{x}{8}-\frac{3}{16}\sin 2x-\frac{1}{16}\left(\sin 2x-\frac{\sin^3 2x}{3}\right)+\frac{3}{16}\left(x+\frac{\sin 4x}{4}\right)+C.$$

□

EXAMPLE 8.2.3 Evaluate $\int \sin^2 x \cos^2 x\,dx$. Use the formulas $\sin^2 x = (1-\cos(2x))/2$ and $\cos^2 x = (1+\cos(2x))/2$ to get:

$$\int \sin^2 x\cos^2 x\,dx = \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,dx.$$

The remainder is left as an exercise. □

Exercises 8.2.

Find the antiderivatives.

1. $\int \sin^2 x\,dx \Rightarrow$

2. $\int \sin^3 x\,dx \Rightarrow$

3. $\int \sin^4 x\,dx \Rightarrow$

4. $\int \cos^2 x \sin^3 x\,dx \Rightarrow$

5. $\int \cos^3 x\,dx \Rightarrow$

6. $\int \sin^2 x\cos^2 x\,dx \Rightarrow$

7. $\int \cos^3 x\sin^2 x\,dx \Rightarrow$

8. $\int \sin x(\cos x)^{3/2}\,dx \Rightarrow$

9. $\int \sec^2 x\csc^2 x\,dx \Rightarrow$

10. $\int \tan^3 x\sec x\,dx \Rightarrow$